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The K_t-Functional for the Interpolation Couple $L_1(A_0)$, $L_\infty(A_1)$

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Let (A_0, A_1) be a compatible couple of Banach spaces in the interpolation theory sense. We give a formula for the K_t-functional of the interpolation couples $(l_1(A_0), c_0(A_1))$ or $(l_1(A_0), l_\infty(A_1))$ and $(L_1(A_0), L_\infty(A_1))$. © 1993 Academic Press, Inc.

We first recall the definition of the K_t-functional which is a fundamental tool in the Lions–Peetre Interpolation Theory and also in Approximation Theory, cf., e.g., [1, 2]. Let (A_0, A_1) be a compatible couple of Banach (or quasi-Banach) spaces. This just means that A_0, A_1 are continuously included into a larger topological vector space (most of the time left implicit), so that we can consider unambiguously the sets $A_0 + A_1$ and $A_0 \cap A_1$. For all $x \in A_0 + A_1$ and for all $t > 0$, we let

$$K_t(x; A_0, A_1) = \inf(\|x_0\|_{A_0} + t\,\|x_1\|_{A_1} \mid x = x_0 + x_1,\ x_0 \in A_0,\ x_1 \in A_1).$$

Recall that the (real interpolation) space $(A_0, A_1)_{\theta, p}$ is defined $(0 < \theta < 1, 1 \leqslant p \leqslant \infty)$ as the space of all x in $A_0 + A_1$ such that $\|x\|_{\theta, p} < \infty$ where

$$\|x\|_{\theta, p} = \left(\int (t^{-\theta} K_t(x; A_0, A_1))^p\, dt/t\right)^{1/p}.$$

It is well known that the K_t-functional for the couple $(L_1(\mu), L_\infty(\mu))$ on a non-atomic measure space (Ω, μ) is given by

$$K_t(f; L_1(\mu), L_\infty(\mu)) = \sup\left\{\int_E |f|\, d\mu,\ E \subset \Omega,\ \mu(E) \leqslant t\right\}.$$

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Let $(\tilde{\Omega}, \tilde{\mu})$ be the measure space obtained by forming the disjoint union of a sequence of copies of (Ω, μ). Since $L_p(\Omega, \mu; l_p)$ can be identified with $L_p(\tilde{\Omega}, \tilde{\mu})$, we have, for all $f=(f_i)$ in $L_1(\mu; l_1)+L_\infty(\mu; l_\infty)$

$$\begin{aligned}
&K_t(f; L_1(\mu; l_1), L_\infty(\mu; l_\infty)) \\
&\quad = \sup\left\{\sum \int_{E_i} |f_i|\, d\mu,\ E_i \subset \Omega, \sum \mu(E_i) \leqslant t\right\} \\
&\quad = \sup\left\{\sum K_{t_i}(f_i; L_1(\mu), L_\infty(\mu)),\ t_i \geqslant 0, \sum t_i \leqslant t\right\}.
\end{aligned}$$

Since $L_p(\mu; l_p)$ and $l_p(\mu))$ can be identified, this example is the prime motivation for the following statement.

THEOREM 1. *Let (A_0, A_1) be a compatible couple of Banach spaces. Consider the pair $(l_1(A_0), l_\infty(A_1))$. Then, $\forall x=(x_i) \in l_1(A_0)+l_\infty(A_1)$, if $x_i=0$ except for finitely many indices, we have*

$$K_t(x; l_1(A_0), l_\infty(A_1)) = \sup\left\{\sum_i K_{t_i}(x_i; A_0, A_1),\ t_i \geqslant 0, \sum t_i \leqslant t\right\}. \quad (1)$$

As a consequence, $\forall x=(x_i) \in l_1(A_0)+c_0(A_1)$, we have

$$K_t(x; l_1(A_0), c_0(A_1)) = \sup\left\{\sum_i K_{t_i}(x_i; A_0, A_1),\ t_i \geqslant 0, \sum t_i \leqslant t\right\}.$$

Proof. Let us denote by C_t the right hand side of the above identity (1). Then it is very easy to check that $C_t \leqslant K_t(x; l_1(A_0), l_\infty(A_1))$. Let us check the converse. Let x be such that $C_t < 1$. This means

$$\sup_{\sum t_i \leqslant t}\left\{\inf_{x_i=a_i+b_i}\left(\sum \|a_i\|_{A_0} + t_i \|b_i\|_{A_1}\right)\right\} < 1. \quad (2)$$

We want to deduce from this the same inequality but with the inf and the sup interchanged. This can be viewed as a consequence of the minimax lemma (which itself is an application of the Hahn–Banach theorem). We prefer to deduce it directly from the Hahn–Banach theorem, as follows. This inequality (2) clearly implies (choosing $t_i = t\xi_i$) that for any non-negative sequence $\xi=(\xi_i)$ such that $\sum \xi_i < 1$ there is, for each index i a decomposition $x_i=\alpha_i+\beta_i$ in A_0+A_1 such that

$$\sum_i \xi_i\left[\left(\sum_k \|\alpha_k\|_{A_0}\right) + t\,\|\beta_i\|_{A_1}\right] < 1. \quad (3)$$

Fix a number $\varepsilon > 0$. We show that the left side of (1) is less than $1+\varepsilon$. We assume that, for some n, we have $x_i = 0$ for all indices $i \geqslant n$. Let $C \subset \mathbf{R}^n$ be the set of all points $y = (y_i)$ of the form

$$y_i = \left(\sum_{k \geqslant 0} \|a_k\|_{A_0}\right) + t\, \|b_i\|_{A_1}, \qquad \text{where} \quad x_i = a_i + b_i,\ a_i \in A_0,\ b_i \in A_1.$$

We claim that the convex hull of C, denoted by $\operatorname{conv}(C)$, intersects $]-\infty, 1+\varepsilon[^n$. Otherwise, by Hahn–Banach (we separate a convex set from an open convex one) we would find a separating functional ξ and a real number r such that $\xi < r$ on $]-\infty, +1]^n$ and $\xi > r$ on C. But (since we obviously can assume $r = 1$) this would contradict (3). This shows that $\operatorname{conv}(C)$ intersects $]-\infty, 1+\varepsilon[^n$, hence we can find decompositions $x_i = a_i^m + b_i^m$, $1 \leqslant m \leqslant M$, and positive scalars $\lambda_1, \ldots, \lambda_m, \ldots, \lambda_M$ with $\sum_m \lambda_m = 1$, such that we have for every index i

$$\sum_m \lambda_m \left[\left(\sum_{k \geqslant 0} \|a_k^m\|_{A_0}\right) + t\, \|b_i^m\|_{A_1}\right] \leqslant 1 + \varepsilon. \tag{4}$$

We can then set

$$a_i = \sum_m \lambda_m a_i^m, \qquad b_i = \sum_m \lambda_m b_i^m.$$

Note that $x_i = a_i + b_i$. Moreover, by (4) and the triangle inequality, for every index i

$$\left(\sum_{k \geqslant 0} \|a_k\|_{A_0}\right) + t\, \|b_i\|_{A_1} \leqslant 1 + \varepsilon,$$

which clearly implies $K_t(x;\, l_1(A_0),\, l_\infty(A_1)) \leqslant 1 + \varepsilon$. By homogeneity, this completes the proof of (1), and the last assertion is immediate. ∎

I asked B. Maurey for some help to extend the preceding statement without unpleasant assumptions and he kindly pointed out to me the following fact and its proof:

THEOREM 2. *Let P_n denote the projection from $l_1(A_0) + l_\infty(A_1)$ onto $l_1(A_0) + l_\infty(A_1)$ which preserves the first n coordinates and annihilates the other ones. Then*

$$\begin{gathered} \forall x \in l_1(A_0) + l_\infty(A_1) \\ K_t(x;\, l_1(A_0),\, l_\infty(A_1)) = \sup_n K_t(P_n(x);\, l_1(A_0),\, l_\infty(A_1)). \end{gathered} \tag{5}$$

Proof. Fix $t>0$. Clearly the right hand side of (5) is not more than its left hand side. Conversely, assume that the right hand side of (5) is <1. We show that the left side also is less than 1. To clarify the notation, if x is a sequence of elements in a Banach space, we denote by $x(k)$ the kth coordinate of x. Then, for all x as in (5) and for all n, there is a decomposition $P_n(x)=x_0^n+x_1^n$ such that

$$\|x_0^n\|_{l_1(A_0)}+t\,\|x_1^n\|_{l_\infty(A_1)}<1. \tag{6}$$

Let $\mathscr{U}$ be a non-trivial ultrafilter on the positive integers. We let n tend to infinity along $\mathscr{U}$ and we denote simply by $\lim_{\mathscr{U}}$ the various resulting limits. Let

$$R=\lim_{\mathscr{U}}\|x_1^n\|_{l_\infty(A_1)} \qquad \text{and} \qquad a_k=\lim_{\mathscr{U}}\|x_0^n(k)\|_{A_0}.$$

Observe that (6) implies

$$\forall K\in\mathbf{N} \qquad \left(\sum_{k<K} a_k\right)+tR\leqslant 1. \tag{7}$$

Now fix $\varepsilon>0$. For each integer k we can find an integer $n_k>k$ large enough so that

$$\|x_0^{n_k}(k)\|_{A_0}<a_k+\varepsilon 2^{-k} \qquad \text{and} \qquad \|x_1^{n_k}\|_{l_\infty(A_1)}<R+\varepsilon.$$

Then we can define

$$x_0(k)=x_0^{n_k}(k) \qquad \text{and} \qquad x_1(k)=x_1^{n_k}(k).$$

Clearly $x(k)=x_0(k)+x_1(k)$ for all k, and moreover

$$\forall K \qquad \sum_{k<K}\|x_0(k)\|_{A_0}+t\sup_{k<K}\|x_1(k)\|_{A_1}<\sum_{k<K}a_k+\varepsilon 2^{-k}+t(R+\varepsilon)$$

hence by (7)

$$\leqslant 1+\varepsilon(2+t).$$

Since this holds for all K, we conclude that $x_0\in l_1(A_0)$, $x_1\in l_\infty(A_1)$, and $\|x_0\|_{l_1(A_0)}+t\,\|x_1\|_{l_\infty(A_1)}\leqslant 1+\varepsilon(2+t)$, and since $\varepsilon>0$ is arbitrary we indeed finally obtain

$$K_t(x;\, l_1(A_0),\, l_\infty(A_1))\leqslant 1. \quad \blacksquare$$

COROLLARY 3. *The formula* (1) *in Theorem* 1 *is valid without any restriction on* $x\in l_1(A_0)+l_\infty(A_1)$.

Remark 4. The formula (1) remains valid with the same proof as above if the spaces A_0 and A_1 are replaced by families of Banach spaces respectively (A_0^n) and (A_1^n). Let us denote by $l_1(\{A_0^n\})$ and $l_\infty(\{A_1^n\})$ the corresponding spaces (these are sometimes called the direct sum of the families (A_0^n) and (A_1^n), respectively, in the sense of l_1 and l_∞). This gives us the following generalized version of (1): for all x in $l_1(\{A_0^n\})+l_\infty(\{A_1^n\})$

$$K_t(x; l_1(\{A_0^n\}), l_\infty(\{A_1^n\})) = \sup\left\{\sum_i K_{t_i}(x_i; A_0^i, A_1^i),\ t_i \geqslant 0, \sum t_i \leqslant t\right\}. \tag{8}$$

We now reformulate our result in the function space case.

THEOREM 5. *Let (A_0, A_1) be a compatible couple of Banach spaces. Let $(\Omega, \mathscr{A}, \mu)$ be an arbitrary measure space. Consider a function f in $L_1(\mu; A_0)+L_\infty(\mu; A_1)$, where we define the Banach space valued L_p-spaces in the Bochner sense. Then, for all $t>0$*

$$K_t(f; L_1(\mu; A_0), L_\infty(\mu; A_1)) = \sup_{\int \phi\, d\mu \leqslant t} \int K_{\phi(\omega)}(f(\omega); A_0, A_1)\, d\mu(\omega), \tag{9}$$

where the sup runs over all non-negative measurable functions ϕ defined on Ω with integral not more than t.

Proof. We may clearly assume that the measure space is σ-finite. Now given a function $f_0 \in L_1(\Omega, \mathscr{A}, \mu; A_0)$, we know (by definition of Bochner measurability, see, e.g., [5, p. 42]) that there is a countable measurable partition of Ω into pieces on each of which the oscillation of f_0 for the norm of A_0 is small. Similarly, given $f_1 \in L_\infty(\Omega, \mathscr{A}, \mu; A_1)$ we know that there is a measurable partition of Ω into pieces on each of which the oscillation of f_1 for the norm of A_1 is small. On the other hand, since the measure space is σ-finite, it admits a countable measurable partition into sets of finite measure, so that, by refining the partitions, we can always assume that the sets have finite measure (so that the conditional expectation makes sense) and that the same partition works for both f_0 and f_1. Consequently, for each $\varepsilon>0$ there is a countable measurable partition of Ω into sets of finite measure on each of which both the A_0-oscillation of f_0 and the A_1-oscillation of f_1 are less than ε. The point of this discussion is the following. Given $f \in L_1(\Omega, \mathscr{A}, \mu; A_0)+L_\infty(\Omega, \mathscr{A}, \mu; A_1)$, we can find a σ-subalgebra $\mathscr{B} \subset \mathscr{A}$ generated by a countable measurable partition of Ω into sets of finite measure such that, if we denote by $f^{\mathscr{B}}$ the conditional expectation of f with respect to $\mathscr{B}$, we have

$$K_t(f-f^{\mathscr{B}}; L_1(\Omega, \mathscr{A}, \mu; A_0), L_\infty(\Omega, \mathscr{A}, \mu; A_1)) < \varepsilon.$$

This reduces the proof of (9) to the case when $\mathscr{A}$ is generated by a countable measurable partition of Ω into sets of finite measure. In that case, we can identify $L_1(\Omega, \mathscr{A}, \mu; A_0)$ and $L_\infty(\Omega, \mathscr{A}, \mu; A_1)$ with suitable sequence spaces and (9) follows easily from (8) (by incorporating the weight of each set of the partition into the norm of the corresponding coordinate). ∎

In the situation of Theorem 5, let us assume (for simplicity) that the intersection $A_0 \cap A_1$ is dense in A_0. Then (cf. [1, p. 303]) we can write for all $x \in A_0 + A_1$

$$K_t(x; A_0, A_1) = \int_0^t k(x, s; A_0, A_1)\, ds,$$

where the k-functional $k(x, s; A_0, A_1)$ is a uniquely defined non-negative, non-increasing, right-continuous function of $s > 0$. In the case of the (scalar valued) couple (L_1, L_∞) over a σ-finite measure space, we find (cf. [1, p. 302])

$$k(x, s; L_1, L_\infty) = x^*(s),$$

where x^* is the non-increasing rearrangement of $|x|$.

Recall the notation $x^{**}(t) = t^{-1} \int_0^t x^*(s)\, ds$, so that $K_t(x; L_1, L_\infty) = tx^{**}(t)$. If $0 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$ we also recall the definition of the quasinorm $\|x\|_{p,q}$ in the Lorentz space $L_{p,q}$ over a σ-finite measure space as

$$\|x\|_{p,q} = \left(\int_0^\infty [t^{1/p} x^*(t)]^q \frac{dt}{t}\right)^{1/q}$$

with the usual convention when $q = \infty$.

If $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, then Hardy's classical inequality shows that this is equivalent to the norm

$$\|x\|_{(p,q)} = \left(\int_0^\infty [t^{1/p} x^{**}(t)]^q \frac{dt}{t}\right)^{1/q}$$

with the usual convention when $q = \infty$. In particular $L_{p,p}$ is the same as L_p with an equivalent norm.

With this notation, we can state

Corollary 6. *In the same situation as Theorem 5, assuming (for simplicity) that the intersection $A_0 \cap A_1$ is dense in A_0, we denote for all f in $L_1(\Omega, \mathscr{A}, \mu; A_0) + L_\infty(\Omega, \mathscr{A}, \mu; A_1)$,*

$$\forall s > 0 \quad \forall \omega \in \Omega \qquad \Psi_f(s, \omega) = k(f(\omega), s; A_0, A_1).$$

Then we have

$$K_t(f;\, L_1(\Omega, \mu;\, A_0),\, L_\infty(\Omega, \mu;\, A_1))$$
$$= K_t(\Psi_f;\, L_1(\Omega \times]0, \infty[,\, d\mu\, ds),\, L_\infty(\Omega \times]0, \infty[,\, d\mu\, ds)). \quad (10)$$

Moreover, for $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, *and* $1/p = 1 - \theta$, *we have*

$$\| f \|_{(L_1(\Omega, \mu;\, A_0),\, L_\infty(\Omega, \mu;\, A_1))_{\theta, q}} = \| \Psi_f \|_{(p, q)}, \quad (11)$$

where the Lorentz space norm is relative to the product space $(\Omega \times]0, \infty[,\, d\mu\, ds)$.

Proof. By (9) we have

$$K_t(\Psi_f;\, L_1(\Omega \times]0, \infty[,\, d\mu\, ds),\, L_\infty(\Omega \times]0, \infty[,\, d\mu\, ds))$$
$$= \sup_{\int \phi\, d\mu \leqslant t} \int K_{\phi(\omega)}(\Psi_f(\cdot, \omega);\, L_1(]0, \infty[,\, ds),\, L_\infty(]0, \infty[,\, ds))\, d\mu(\omega)$$

using (9) again this yields (10) since we have obviously

$$\forall t > 0,\ \forall \omega \in \Omega \qquad K_t(\Psi_f(\cdot, \omega);\, L_1([0, \infty[,\, ds),\, L_\infty(]0, \infty[,\, ds))$$
$$= \int_0^t \Psi_f(s, \omega)\, ds = K_t(f(\omega);\, A_0, A_1).$$

Clearly (11) is an immediate consequence of (10) by applying $K_t(x;\, L_1, L_\infty) = t x^{**}(t)$ on the product space with $x = \Psi_f$. ∎

Remark 7. As an application of Corollary 6, one can derive the well known Lions–Peetre results on interpolation between vector valued L_p-spaces in a rather transparent way, for example, in the situation of Corollary 6, if $q = p$ and $1/p = 1 - \theta$, we have

$$(L_1(\Omega, \mathscr{A}, \mu;\, A_0),\, L_\infty(\Omega, \mathscr{A}, \mu;\, A_1))_{\theta, p} = L_p(\Omega, \mathscr{A}, \mu;\, (A_0, A_1)_{\theta, p}).$$

Indeed, when $p = q > 1$ Hardy's classical inequality (see [1, pp. 124 and 219]) shows that for all x in $A_0 + A_1$, $\|k(x, s;\, A_0, A_1)\|_{L_p(ds)}$ is equivalent to the norm of x in $(A_0, A_1)_{\theta, p}$. Therefore, since $\|\Psi_f\|_{(p,p)}$ is equivalent to $\|\Psi_f\|_{L_p(d\mu\, ds)}$, it is equivalent to the norm of f in $L_p(\Omega, \mathscr{A}, \mu;\, (A_0, A_1)_{\theta, p})$.

In fact, one finds more generally that if $1/p = 1 - \theta$ then for all $1 \leqslant q \leqslant p$ the following well known inclusion holds

$$(L_1(\Omega, \mathscr{A}, \mu;\, A_0),\, L_\infty(\Omega, \mathscr{A}, \mu;\, A_1))_{\theta, q} \subset L_p(\Omega, \mathscr{A}, \mu;\, (A_0, A_1)_{\theta, q}).$$

Moreover, when $q \geqslant p$ the reverse inclusion holds. We refer to [4] for counterexamples to the other inclusions.

Remarks. (i) Using the "power theorem" (cf. [2, p. 68]) it is easy to deduce from Theorem 5 an equivalent of the K_t-functional for the couple $L_p(\Omega, \mathscr{A}, \mu; A_0)$, $L_\infty(\Omega, \mathscr{A}, \mu; A_1)$ for $0<p<\infty$, when (A_0, A_1) are Banach spaces.

(ii) More generally, if $1 \leqslant p_0, p_1 < \infty$ then there are simple natural quantities known to be equivalent to the K_t-functional for the couple $L_{p_0}(\Omega, \mathscr{A}, \mu; A_0)$, $L_{p_1}(\Omega, \mathscr{A}, \mu; A_1)$. In the case p_1 is finite, these can be derived easily from the trivial case $p_0 = p_1$ and the power theorem, and this argument even works when (A_0, A_1) are quasi-Banach spaces. This application of the power theorem was pointed out to me by Quanhua Xu, but Cwikel informed me that this was already known to J. Peetre (cf. also [8]). Apparently however this approach does not yield the case $p_1 = \infty$ which is the main point of the present paper.

We give as an application a generalization of an embedding theorem for L_p spaces, namely the following. If $(\Omega', \mathscr{A}', \mu')$ is an arbitrary measure space, we can define a linear operator

$$T_p\colon L_p(\Omega', \mu') \to L_{p,\infty}(\Omega' \times \left]0, \infty\right[, d\mu'\, ds)$$

as follows (here $0<p<\infty$ and we intentionally denote below by ω a positive real number instead of s and change the notation ds to $d\omega$)

$$\forall f \in L_p(\Omega', \mu') \qquad T_p(f)(\omega', \omega) = \omega^{-1/p} f(\omega').$$

Then it is a simple exercise to check that T_p is an isometric embedding, i.e., we have

$$\forall f \in L_p(\Omega', \mu') \qquad \|T_p(f)\|_{p,\infty} = \|f\|_p. \tag{12}$$

Actually, if we denote by m the product measure $dm = d\mu' \times d\omega$, we have

$$\forall t>0 \qquad t^p m(\{|T_p(f)| > t\}) = \int |f|^p\, d\mu'. \tag{13}$$

Similarly, let us denote by ν the counting measure on the set $\mathbf{N}^*$ of all positive integers. Then the preceding embedding has the following discrete counterpart. We define a linear operator

$$S_p\colon L_p(\Omega', \mu') \to L_{p,\infty}(\Omega' \times \mathbf{N}^*, d\mu'\, d\nu)$$

as follows $(0<p<\infty)$

$$\forall f \in L_p(\Omega', \mu') \qquad S_p(f)(\omega', n) = n^{-1/p} f(\omega').$$

Again, it is easy to check that

$$\forall f \in L_p(\Omega', \mu') \qquad \|S_p(f)\|_{p,\infty} = \|f\|_p.$$

Moreover, if we denote, for any positive real r, by $[r]$ the largest integer $n < r$, and if we denote by m' the product measure $dm' = d\mu' \times dv$, we clearly have

$$\forall t > 0 \qquad m'(\{|S_p(f)| > t\}) = \int \left[\frac{|f|^p}{t^p}\right] d\mu'.$$

We now return to the abstract case

THEOREM 8. *In the same situation as Theorem 5, assume (for simplicity) that the intersection $A_0 \cap A_1$ is dense in A_0, and let $1 < p < \infty$, $\theta = 1 - 1/p$. We define more generally two linear operators*

$$T_p\colon (A_0, A_1)_{\theta,p} \to (L_1(]0, \infty[, d\omega; A_0), L_\infty(]0, \infty[, d\omega; A_1))_{\theta,\infty}$$

$$S_p\colon (A_0, A_1)_{\theta,p} \to (L_1(\mathbf{N}^*, dv; A_0), L_\infty(\mathbf{N}^*, dv; A_1))_{\theta,\infty}$$
$$= (l_1(A_0), l_\infty(A_1))_{\theta,\infty}$$

by setting

$$\forall x \in (A_0, A_1)_{\theta,p} \qquad T_p(x) = (\omega \to \omega^{-1/p} x) \qquad \textit{and} \qquad S_p(x) = (n \to n^{-1/p} x).$$

Then we have $\forall x \in (A_0, A_1)_{\theta,p}$

$$\|T_p(x)\|_{(L_1(]0,\infty[,d\omega;A_0),L_\infty(]0,\infty[,d\omega;A_1))_{\theta,\infty}} = p'\left(\int_0^\infty k(x, s; A_0, A_1)^p\, ds\right)^{1/p}. \tag{14}$$

Therefore (by Hardy's inequality), T_p is an isomorphic embedding. Similarly, S_p is an isomorphic embedding.

Proof. Let $f(\omega) = \omega^{-1/p} x$. Then we have

$$\Psi_f(s, \omega) = \omega^{-1/p} k(x, s; A_0, A_1).$$

Note that by (13) we have

$$\forall t > 0 \qquad \Psi_f^*(t) = t^{-1/p}\left(\int_0^\infty k(x, s; A_0, A_1)^p\, ds\right)^{1/p}.$$

Hence $\Psi_f^{**}(t) = p' t^{-1/p}(\int_0^\infty k(x, s; A_0, A_1)^p\, ds)^{1/p}$ and (14) follows from (11) with $q = \infty$. The discrete case is now easy and left to the reader. ∎

Remark 9. We do not see how to completely extend the preceding facts in the case of quasi-Banach spaces A_0, A_1, with $r<1$ and with $L_r(A_0)$ instead of $L_1(A_0)$. However, the easy direction in Theorems 1 or 5 obviously extends up to a constant. For instance, there is a constant c such that $\forall x \in l_r(A_0)+l_\infty(A_1)$ and $\forall t>0$

$$\sup_{\sum t_i^r \leqslant t^r} \left(\sum K_{t_i}(x_i;\, A_0, A_1)^r\right)^{1/r} \leqslant cK_t(x;\, l_r(A_0), l_\infty(A_1)). \tag{15}$$

To illustrate the possible uses of Theorem 8, we conclude by an application to the complex interpolation method which develops in a more abstract way an idea presented in [9] in the context of H^p spaces. Again, let (A_0, A_1) be a compatible couple of Banach spaces included in a topological vector space V. Assume moreover that there is a quasi-Banach space B also included in V and such that for some $0<a<1$ we have

$$A_0=(B, A_1)_{a,1}.$$

Let $r=1-a$. As a typical example of this situation the reader should think of $B=L_r$, $A_0=L_1$, $A_1=L_\infty$. For any $x \in A_0+A_1$, we denote by $S^0(x)$ the sequence $(x/n)_{n>0}$ and more generally for any complex number z we denote by $S^z(x)$ the sequence $(x/n^{1-z})_{n>0}$. Moreover we make the rather restrictive assumption that S^0 defines a bounded operator from A_0 into $(l_r(B), l_\infty(A_1))_{a,\infty}$. The reader will easily check (using (12) and (13) above) that this holds for the preceding example with $B=L_r$. Then we claim that there is a bounded inclusion mapping

$$\forall 0<\theta<1 \qquad (A_0, A_1)_\theta \subset (A_0, A_1)_{\theta,p} \qquad \text{if} \quad \frac{1}{p}=1-\theta. \tag{16}$$

See [7] for a somewhat related result. Let us sketch the proof of (16). Consider an element x in the open unit ball of the space $(A_0, A_1)_\theta$. Then there is an analytic function f with values in A_0+A_1 on the strip $0<\Re(z)<1$, which is continuous in the closed strip, such that $f(\theta)=x$ and such that for all real numbers t, $f(it)$ is in the unit ball of A_0 and $f(1+it)$ is in the unit ball of A_1 (and their respective norms tend to zero when t tends to infinity). We now apply Stein's interpolation principle to the analytic family of operator S^z. Consider $g(z)=S^z f(z)$. Note that $g(\theta)=S_p(x)$. For simplicity, let us denote $C=(l_r(B), l_\infty(A_1))_{\alpha,\infty}$. By our restrictive assumption we have $\sup_t \|g(it)\|_C \leqslant c_0$ (where c_0, c_1, c_2, etc., are constants) and trivially we have $\sup_t \|g(1+it)\|_{l_\infty(A_1)} \leqslant 1$. Therefore, we obtain $\|g(\theta)\|_{C,l_r(A_1))_\theta} \leqslant c_1$. Since $(C, l_\infty(A_1))_\theta \subset (C, l_\infty(A_1))_{\theta,\infty}$, we deduce from the reiteration principle (cf. [2, p. 48]) that if $b=(1-\theta)a+\theta$ we have $\|g(\theta)\|_{l_r(B),l_\infty(A_1))_{h,\infty}} \leqslant c_2$. By Remark 9 and the same computations as above we have

$$\|x\|_{(A_0,A_1)_{\theta,p}} \leqslant c_3 \|S_p(x)\|_{(l_r(B),l_\infty(A_1))_{h,\infty}},$$

so that (recalling $g(\theta)=S_p(x)$) we finally find $\|x\|_{(A_0,A_1)_{\theta,p}} \leqslant c_4$. This concludes the proof of the above claim (16). (The reader should easily fill the minor technical gaps that we left to avoid obscuring the idea.)

Now assume given a closed subspace $S \subset V$ and let

$$S_0 = S \cap A_0, \qquad S_1 = S \cap A_1, \qquad \beta = S \cap B.$$

Let $Q_0 = A_0/S_0$, $Q_1 = A_1/S_1$, and $Q = B/\beta$ be the associated quotient spaces. Clearly (Q_0, Q_1) form a compatible couple since there are natural inclusion maps

$$Q_0 \to V/S \qquad \text{and} \qquad Q_1 \to V/S,$$

and similarly $Q \to V/S$. Obviously, after composition with the quotient mappings in the above assumption, we get a bounded map from A_0 into $(Q, Q_1)_{a,1}$, hence (since the latter vanishes on S_0) we have a bounded map from Q_0 into $(Q, Q_1)_{a,1}$. Similarly, we find that the same restrictive assumption as above is satisfied by the quotient spaces and therefore we conclude that

$$\forall 0<\theta<1 \qquad (Q_0, Q_1)_\theta \subset (Q_0, Q_1)_{\theta,p} \qquad \text{if } \frac{1}{p} = 1-\theta. \tag{17}$$

An alternative to the above restrictive assumption is to assume the following: there is a Banach space $D \subset (A_0 + A_1)^{\mathbf{N}^*}$ and a constant c such that

$$\forall x \in A_0, \forall t \in \mathbf{R} \qquad \|S^{it}x\|_D \leqslant c\,\|x\|_{A_0}, \tag{18}$$

and

$$(D, l_\infty(A_1))_{\theta,\infty} \subset (l_1(A_0), l_\infty(A_1))_{\theta,\infty}. \tag{19}$$

Then (16) holds. Indeed with the same notation as above, if $\|x\|_{(A_0,A_1)_\theta} < 1$, this gives $\|g(\theta)\|_{(D,l_\infty(A_1))_\theta} \leqslant c_1$, hence a fortiori $\|g(\theta)\|_{(D,l_\infty(A_1))_{\theta,\infty}} \leqslant c_2$, therefore by (19), $\|S_p(x)\|_{(l_1(A_0),l_\infty(A_1))_{\theta,\infty}} \leqslant c_3$, and by Theorem 8, finally $\|x\|_{(A_0,A_1)_{\theta,p}} \leqslant c_4$. The assumptions (18) and (19) are slightly more general than the preceding one but seem less easy to verify in practice.

In [9], the preceding argument is applied in the particular case $A_0 = L_1$, $Q_0 = L_1/H^1$, $A_1 = L_\infty$, $Q_1 = L_\infty/H^\infty$ to give a new proof that (17) holds in this case, which is originally due to Peter Jones [6]. We refer the reader to [9] for more information on this topic. Concerning for instance H^p-spaces with several complex variables or Sobolev spaces on $\mathbf{R}^n$ (cf. Bourgain's recent paper [3]) the preceding remarks show that whenever the appropriate real interpolation results hold, the corresponding complex interpolation results will also hold. Unfortunately, the real interpolation results do not seem complete enough at the moment to yield the assumptions needed in the above remarks.

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